Bounds on Information and the Security of Quantum Cryptography

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Abstract

Strong attacks against quantum key distribution use quantum memories and quantum gates to attack directly the final key. In this paper we extend a novel security result recently obtained, to demonstrate proofs of security against a wide class of such attacks. To reach this goal we calculate information-dependent reduced density matrices, we study the geometry of quantum mixed state, and we find bounds on the information leaked to an eavesdropper. Our result suggests that quantum cryptography is ultimately secure.

Quantum cryptography (e.g. [1, 2]) suggests an *information secure* key distribution. It is based on the fact that non-orthogonal quantum states cannot be cloned, and any attempt to obtain information regarding these states necessarily disturbs them and induces noise. In principle, the legitimate users of a quantum key distribution scheme, Alice and Bob, should quit the protocol if they notice a noise. However, in real protocols, the channels and devices are not perfect, and some errors are inevitable. As long as the rate of errors is small, these errors must be accepted and corrected by the legitimate users. As a result, the eavesdropper, Eve, can obtain some information on the transmitted data, as long as she induces less errors than allowed (e.g., by eavesdropping on a small portion of the transmitted particles). Furthermore,

she can obtain more information using the error-correction data transmitted via a classical channel.

To overcome these problems, privacy amplification techniques [3] were suggested. The simplest technique uses the parity bit of a long string as the secret bit (where the parity is zero if the string contains an even number of 1's and else it is one). Such techniques aim to reduce Eve's information on the final key to be exponentially small with the length of the initial string (or at least to be much smaller than a single bit). Unfortunately, a proof of security must stand against an adversary equipped with *any* technology allowed by the rules of quantum mechanics, and neither of the suggested schemes is proven secure (for a different opinion see [4]); their security against sophisticated *joint* attacks, which use quantum memories, quantum gates, and delayed measurements to attack *directly* the final key, is only partially established [5, 6, 7]. In this work we extend the results of [7] much further.

The first hints that privacy amplification might still be effective against such attacks were provided by Bennett, Mor and Smolin (BMS) [8]. Suppose Eve obtains a binary string of n bits where each bit is presented by non-orthogonal polarization states, $\psi_0 = \binom{\cos\alpha}{\sin\alpha}$ or $\psi_1 = \binom{\cos\alpha}{-\sin\alpha}$, with *small* angle 2α between them. If each bit is measured separately, the optimal information on the parity bit of the string, $I_S(n,\alpha) = (2\alpha)^{2n}/(2\ln 2)$, is exponentially small with n. By measuring all particles together Eve can gain much more information on the parity bit. However, the optimal information, $I_M(n,\alpha) = c\binom{2k}{k}\alpha^{2k}$ (with $n = 2k$ and $c = 1$ for even n, and $n = 2k-1$ and $c = 1/\ln 2$ for odd n), is still exponentially small with the length of the string. This result (henceforth, the BMS result) suggests that quantum cryptography is secure even when Eve knows the specification of the privacy amplification technique, since all privacy amplification techniques are based on similar principles.

In real protocols, Eve does not obtain one of two states with small angle between them, but she can probe the states sent from Alice to Bob using any technique she likes. Thus, the BMS result only provides some intuition regarding the effectiveness of privacy amplification. To make this intuition more adequate to realistic quantum key distribution protocols, Biham and Mor [7] presented a restricted class of joint attacks, called *collective attacks*, which can use the BMS method and result: (a) Eve attaches a <u>separate, uncorrelated</u> probe to each transmitted particle using a translu-

cent attack; (b) Eve keeps the probes in a quantum memory till receiving all classical data including error-correction code and privacy amplification data; (c) Eve performs the optimal measurement on her probes in order to learn the optimal information on the final key. The underlined constraints on the probes distinguish the collective attacks from more general joint attacks, and enable analyzing the attacks in terms of the density matrices which Eve obtains.

We concentrate on *symmetric collective attacks* in which the same translucent attack is applied to each transmitted particle, and the attack is symmetric to any of the allowed quantum states of each particle. Such an attack induces the same probability of error to each transmitted bit. It must be weak, or else it would induce a non acceptable error-rate. Thus, the possible states of Eve's probe cannot differ much.

An explicit example of such a symmetric collective attack was presented in [7], together with a proof of security against it. In this example Alice and Bob use the two state scheme of Bennett [2] (with pure polarization states with angle 2θ between them). Eve uses, in the first step of the collective attack, the (weak) translucent attack without entanglement [9] (which we call here the EHPP attack), that leaves each probe in one of two pure states, ψ_0 or ψ_1, with small angle 2α between them. After an error-estimation step, Alice and Bob have an n-bit string. Alice and Bob choose the parity bit of that (full n-bit) string to be their secret bit, and Alice sends to Bob some parities of substrings as the error-correction data[10]. In [7] we calculated Eve's density matrices for the parity bit while taking into account the error-correction data she has [11]. Then, we found Eve's best strategy for measuring the probes and her optimal mutual information on the parity bit. For large strings and small error-rate (thus, small angle α) this information decreases exponentially with the length of the string n; e.g., for Hamming codes it is

$$I(n,\alpha) \leq C(n)(2\alpha)^{(n+1)/2} \ , \tag{1}$$

with $C(n) = \frac{2}{\ln 2\sqrt{\pi}}\sqrt{(n+1)}$. For a given error-rate, p_e, the resultant angle in the EHPP attack [12] is $\alpha = (\tan^2(2\theta)\ p_e)^{1/4}$, so that the information $I(n,p_e)$ is of the order of $p_e^{(n+1)/8}$.

In terms of quantum information theory this result (henceforth, the BM result) extends the BMS result to the case where parities of substrings are given (error-correction code). For purposes of quantum key distribution, the

BM result provides the first security proof against a strong attack. However, it is restricted to attacks in which Eve's probes are in a pure state. Unfortunately, most possible translucent attacks on the two state scheme [2], which can be used in the first step of the collective attack, leave each of Eve's probes in a mixed state. Also, any translucent attack on the four state scheme [1] leaves each probe in a mixed state (at least for two out of the four possible states).

The aim of this work is to apply the BM result to the case of mixed states. We first demonstrate that any type of information which can be extracted from certain two-dimensional mixed states can be bounded, if the solution for pure states is known. Then we explicitly demonstrate, via two examples, how to bound Eve's optimal information (for a given induced error-rate). We also calculate the (individual bit) *information-dependent* reduced density matrices which are in Eve's hands.

Any state (density matrix) in 2-dimensional Hilbert space can be written as $\rho = \frac{\hat{I}+r\cdot\hat{\sigma}}{2}$ so that $\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$, with $r = (x, y, z)$ being a vector in $\mathcal{R}^3$, $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ the Pauli matrices, and $\hat{I}$ the unit matrix. In this "spin" notations, each state is represented by the corresponding vector r. For pure states $|r| = 1$, and for mixed states $|r| < 1$. Suppose that χ and ζ are two density matrices, represented by r_χ and r_ζ respectively. It is possible to construct the density matrix $\rho = m\zeta + (1-m)\chi$ from the two matrices (where $0 \leq m \leq 1$), and the geometric representation of such a density matrix $\rho = \frac{\hat{I}+r_\rho\cdot\hat{\sigma}}{2}$ is $r_\rho = mr_\zeta + (1-m)r_\chi$. Two pure states can always be expressed as $|\Phi_0\rangle = \binom{c}{s}$ and $|\Phi_1\rangle = \binom{c}{-s}$, with $c = \cos\alpha$ and $s = \sin\alpha$. Using the notations of density matrices ($\Phi_0 \equiv |\Phi_0\rangle\langle\Phi_0|$ etc.) the two pure states are $\Phi_p = \frac{1}{2}\begin{pmatrix} 1+z & x \\ x & 1-z \end{pmatrix}$, with $z = \cos 2\alpha$ and $x = \sin 2\alpha$ for $p = 0$ and $x = -\sin 2\alpha$ for $p = 1$. If Φ_p is used to describe a bit p, the receiver can identify the bit by distinguishing the two pure states. Two (not necessarily pure) density matrices ρ_p in two-dimensional Hilbert space, with equal determinants (which are equal to $|r|$) can also be expressed using similar form with $z = |r|\cos 2\alpha$ and $x = \pm|r|\sin 2\alpha$. For two such mixed states let us choose a state χ_n, and two pure states Φ_0, Φ_1 such that

$$\rho_0 \quad = \quad m\Phi_0 + (1-m)\chi_n$$

$$\rho_1 \;=\; m\Phi_1 + (1-m)\chi_n \;. \tag{2}$$

Let I be some (positive) measure for the optimal distinguishability of two states, so that *any operation done on them* cannot lead to a distinguishability better than I. From the construction of (2), it is clear that any measure for optimal distinguishability must find that the two mixed states ρ_p are not more distinguishable than the two pure states Φ_p [that is $I(\Phi_0;\Phi_1) \geq I(\rho_0;\rho_1)$]: Suppose the contrary $I(\Phi_0;\Phi_1) < I(\rho_0;\rho_1)$. Then, when one receives Φ_p he can mix them with χ_n and derive a better distinguishability than $I(\Phi_0;\Phi_1)$, in contradiction to the definition of $I(\Phi_0;\Phi_1)$.

We can choose any measure of an optimal information carried by these systems to describe the distinguishability, and it should satisfy $I_{mixed} \leq I_{pure}$. Very complicated types of information can be extracted from such systems, as for example, the optimal information on the parity of an n-bit string of such quantum bits [8, 7]. In the case [7], where parities of substrings are given, a solution exists only for pure state with small angles (the BM result), and we can now use this known solution to bound the optimal information which can be extracted from mixed states which are close to each other. Let $\rho_{\rm cms}$ be the completely mixed state $\rho_{\rm cms} = \frac{1}{2}\hat{I}$. Also let $\rho_\downarrow$ be the pure state of spin down in the z direction. Two cases of eq. (2) are useful for our purpose: (a) $\rho_p = m\Phi_p + (1-m)\rho_{\rm cms}$, where the pure states Φ_p have the same angle as ρ_p (see fig. 1a); (b) $\rho_p = m\Phi_p + (1-m)\rho_\downarrow$, where Φ_p (which are uniquely determined) are shown in Fig.1b. The first type of bound is useful if ρ_p have a small angle α (which satisfies $\tan 2\alpha = x/z$), so that the angle between the pure states $\beta = \alpha$ is also small. The second type of bound is useful when the 'distance' $2x$ between the two possible mixed states is small (while α might be large). In this case x is small and z positive hence the resultant angle β between the two pure states is small (following $\tan\beta = \tan 2\delta = \frac{x}{z+1} \leq x$). Thus, in both cases the angle between the two pure states is small so that $I(n,\beta)$, eq. (1) with an angle β, provides an upper bound on Eve's information on the final key.

An explicit calculation of Eve's density matrix as a function of p_e must be done separately for any suggested attack to obtain $I(n,p_e)$. However, the fact that Eve is allowed to induce only small error-rate restricts her possible transformations at the first stage of the collective attack, hence, the two possible states of each of her probes must be largely overlapping (for a symmetric attack). Concentrating on two-dimensional probes, this

promises us that the second of the two cases above can *always* be used to bound Eve's information to be exponentially small. For certain examples – the first case is sufficient, hence the angle between the two possible pure states can be calculated from Eve's density matrix directly using $\beta = \alpha$. Let us show two examples in details, to conclude that Eve's information is exponentially small with the length of the string. Both examples use the same unitary transformation but are applied onto different quantum cryptographic schemes, the two state scheme [2] and the four state scheme [1].

In our examples Eve uses a 2-dimensional probe in an initial state $\binom{1}{0}$. She performs a unitary transformation $U\binom{1}{0}|\phi\rangle$ (with $|\phi\rangle$ Alice's state), where

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma & 0 \\ 0 & s_\gamma & c_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{3}$$

with $c_\gamma = \cos\gamma$, etc. She chooses a small angle γ so that the attack is weak. Let Alice's possible initial states be $|\phi_p\rangle = \binom{\cos\theta}{\pm\sin\theta}$ in the two state scheme, and $|\phi_m\rangle = \frac{1}{\sqrt{2}}\binom{1}{i^m}$ (with $m = 0\cdots 3$) in the four state scheme. The corresponding final states are

$$|\Psi_p\rangle = \begin{pmatrix} \cos\theta \\ \pm\sin\theta c_\gamma \\ \pm\sin\theta s_\gamma \\ 0 \end{pmatrix} \quad ; \quad |\Psi_m\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i^m c_\gamma \\ i^m s_\gamma \\ 0 \end{pmatrix} , \tag{4}$$

respectively. Bob's reduced density matrices (rdms) are calculated from $|\Psi\rangle\langle\Psi|$ by tracing out Eve's particle. This operation is usually denoted by $\rho_B = \mathrm{Tr}_E\, [|\Psi\rangle\langle\Psi|]$, where the full formula is given by eq. 5.19 in [13] ($\rho_{nm} = \sum_{\mu\nu} \rho_{n\nu,m\mu}\delta_{\mu\nu} = \sum_\mu \rho_{n\mu,m\mu}$). We denote this operation by $\rho_B = \mathrm{Tr}_E\, \left[(|\Psi\rangle\langle\Psi|)\hat{I}\right]$, where $\hat{I}$ is two dimensional ($\delta_{\mu\nu}$ in eq. 5.19). From Bob's matrices we find the error-rate, that is, the probability p_e that he recognizes a wrong bit value. Calculating Eve's density matrix is more tricky; we need to take into account the additional information she obtains from the classical data, in order to obtain an information-dependent rdms. This is a trivial task for the four-state scheme but a rather confusing task in case of the two-state scheme.

In case of the four state scheme Bob measures his particle in one of the basis x (corresponding to $m = 0, 2$) or y ($m = 1, 3$). Suppose that Alice and Bob use the x basis; Bob's rdms are $\rho_B = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(s_\gamma)^2 & \pm\frac{1}{2}c_\gamma \\ \pm\frac{1}{2}c_\gamma & \frac{1}{2} - \frac{1}{2}(s_\gamma)^2 \end{pmatrix}$, leading to an error-rate $p_e = \sin^2(\gamma/2)$ which is the probability that he identifies $|\phi_2\rangle$ when $|\phi_0\rangle$ is sent. Eve has the same knowledge of the basis, hence her information-dependent rdms are $\rho_E = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(c_\gamma)^2 & \pm\frac{1}{2}s_\gamma \\ \pm\frac{1}{2}s_\gamma & \frac{1}{2} - \frac{1}{2}(c_\gamma)^2 \end{pmatrix}$, so that $x = s_\gamma$, $z = (c_\gamma)^2$, and the relevant angles are $2\beta = 2\alpha = (\tan)^{-1}(s_\gamma/{c_\gamma}^2)$ (using the first type of bounds). For a small angle γ we get $p_e \approx \gamma^2/4 + O(\gamma^4)$, $\beta \approx \gamma/2 + O(\gamma^3)$, and thus $p_e \approx \beta^2 + O(\beta^4)$. The information is thus bounded by $I(n, p_e) < C(n)(4p_e)^{(n+1)/4}$ to be exponentially small [using eq. (1)].

In case of the two-state scheme Bob's rdms are $\rho_B = \begin{pmatrix} (c_\theta)^2 + (s_\theta)^2(s_\gamma)^2 & \pm c_\theta s_\theta c_\gamma \\ \pm c_\theta s_\theta c_\gamma & (s_\theta)^2(c_\gamma)^2 \end{pmatrix}$. Bob chooses one of two possible measurements, $M_{0\to1}$ or $M_{1\to0}$, with equal probability $p_{0\to1} = p_{1\to0} = 1/2$; In case of $M_{0\to1}$, Bob measures the received state to distinguish ϕ_0 from its orthogonal state ${\phi_0}'$ and finds a conclusive result '1' whenever he gets ${\phi_0}'$. (The conclusive result '0' is obtained by replacing 0 and 1 in the above). The error-rate is the probability of identifying ${\phi_p}'$ when ϕ_p is sent, and it is $p_e = (s_\theta)^2(c_\theta)^2[1 - c_\gamma]^2 + (s_\theta)^4(s_\gamma)^2$.

To obtain Eve's density matrices one must take into account all the information she possibly has. If one ignores the classical information and calculates the standard rdms (as in [14]), then the result is of significant importance to quantum information, while it is less relevant to quantum cryptography. Recall that Bob keeps only particles identified conclusively (as either ${\phi_0}'$ or ${\phi_1}'$); Bob informs Alice — and thus Eve — which they are, and, as a result, Eve knows that Bob received either ${\phi_0}'$ or ${\phi_1}'$ in his measurement, and not ϕ_0 or ϕ_1. This fact influences her density matrices, and these are not given anymore by the simple tracing formula $\rho_E = \mathrm{Tr}_B \left[(|\Psi\rangle\langle\Psi|)\hat{I}\right]$. In general, *information dependent* rdms are obtained by replacing $\hat{I}$ by any other positive operator $\hat{A}$:

$$\rho_E = \mathrm{Tr}_B \left[(|\Psi\rangle\langle\Psi|)\hat{A}\right] \tag{5}$$

(This is a rather obvious conclusion from the discussions prior to eq. 5.19 and also from page 289 in sec. 9 in [13]; The correctness of this technique can easily be verified [15]). In our case $\rho_E = \mathrm{Tr}_B \left[(|\Psi\rangle\langle\Psi|)(\frac{1}{2}|{\phi_0}'\rangle\langle{\phi_0}'| + \frac{1}{2}|{\phi_1}'\rangle\langle{\phi_1}'|)\right]$,

where the halves result from $p_{0\to 1}$ and $p_{1\to 0}$. This tracing technique leads to

$$\rho_E = \begin{pmatrix} (s_\theta)^2(c_\theta)^2 + (s_\theta)^2(c_\theta)^2(c_\gamma)^2 & \pm c_\theta(s_\theta)^3 s_\gamma \\ \pm c_\theta(s_\theta)^3 s_\gamma & (s_\theta)^4(s_\gamma)^2 \end{pmatrix}. \tag{6}$$

After normalization we get $x = 2s_\gamma c_\theta(s_\theta)^3/\mathrm{Tr}\rho_E$ and $z = \frac{1+z}{2} - \frac{1-z}{2} = [(c_\theta)^2(s_\theta)^2[1+(c_\gamma)^2] - (s_\theta)^4(s_\gamma)^2)/\mathrm{Tr}\rho_E$. The relevant angles are again $2\beta = 2\alpha = \tan^{-1}(x/z)$. For small angle γ we get $p_e \approx {s_\theta}^4\gamma^2 + O(\gamma^4)$, $2\beta \approx (s_\theta/c_\theta)\gamma + O(\gamma^3)$. Finally we get $p_e \approx (s_\theta)^2(c_\theta)^2(2\beta)^2 + O(\beta^4)$ from which $I(p_e, n)$ can be easily calculated as in the previous example.

The information available to Eve when she performs any other symmetric collective attack with two-dimensional probes can also be calculated using our method. Although we do not know to find the optimal attack of that type yet, our method can still prove security against it, since there is some (small enough) error-rate, such that Eve's probes have small angles between them, and thus, our proof can be applied. The second type of bounds is usually irrelevant when the attack is given (since Eve's initial state is usually in pure state), but it can be very useful for finding the optimal attack, requiring only to find the maximal 'distance', $2x$, between the two possible states of the probe.

More general collective attacks can use non-symmetric translucent attacks and/or can use probes in higher dimensions, in the first step of the collective attack. Methods similar to ours can be used for proving security against various non-symmetric collective attacks (in 2-dimensions), but the calculation becomes more complicated and is beyond the goals of this work. Our bounds cannot be used when Eve uses higher dimensional probes. Indeed, in this case the two possible states of each probe are still highly overlapping, and the same intuition which holds in our paper shall still hold. However, extending the information bounds we found to three or four dimensions might be a difficult task (such analysis of dimensions higher than four is not required since they cannot help the attacker due to the reasons shown in [14]).

A more crucial issue is the possibility of finding stronger joint attacks which are not collective. Let us present the strong argument which is the basis for approaching the security problem through the collective attack: by the time Eve holds the transmitted particles she has no knowledge of the error-correction and privacy amplification techniques to be used by Alice and Bob. She even doesn't know which particles will be discarded in the

error-estimation stage, and how the common bits will be reordered. Thus, we conjecture that she cannot gain information by searching or by creating correlations between the transmitted particles; she better keep one separate probe for each particle, and perform the measurements after obtaining the missing information as is done in the collective attacks. Any attempt of creating such coherent correlations at the first step of the attack induces error, while it cannot lead to an increase in the resultant information; indeed it could help Eve if she would guess correctly the required correlations (e.g., the final string, from which the parity is calculated), but the probability of successful guess is exponentially small. Unfortunately, proving this intuitive argument is yet an open problem.

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References

[1] C. H. Bennett and G. Brassard, in *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, New York, 1984) p. 175.

[2] C. H. Bennett, *Phys. Rev. Lett.* **68**, 3121 (1992).

[3] C. H. Bennett, G. Brassard, C. Crépeau and U. Maurer, *IEEE Trans. Info. Theo.* **41**, 1915 (1995).

[4] D. Mayers, *Proc. of Crypto 96, LNCS* **1109**, 343 (1996).

[5] A. Yao, *Proc. 26 Symp on the Theo. of Comp.*, 67 (1995), proved security in case of error free channels.

[6] D. Deutsch et al, *Phys. Rev. Lett.* **77**, 2818 (1996), proved security in case of noisy channel and error free devices.

[7] E. Biham and T. Mor, "On the Security of Quantum Cryptography Against Collective Attacks", *to be published in Phys. Rev. Lett.*, quant-ph 9605007.

[8] C. H. Bennett, T. Mor and J. Smolin, *Phys. Rev. A* **54**, 2675 (1996).

[9] A.K. Ekert, B. Huttner, G.M. Palma and A. Peres, *Phys. Rev. A.* **50**, 1047 (1994).

[10] The analysis suggested in [7] applies only when Alice sends error-correction data to Bob. It does not apply if bits are thrown away as in the error-correction of Bennett et al, *j. Crypt.* **5**, 3 (1992).

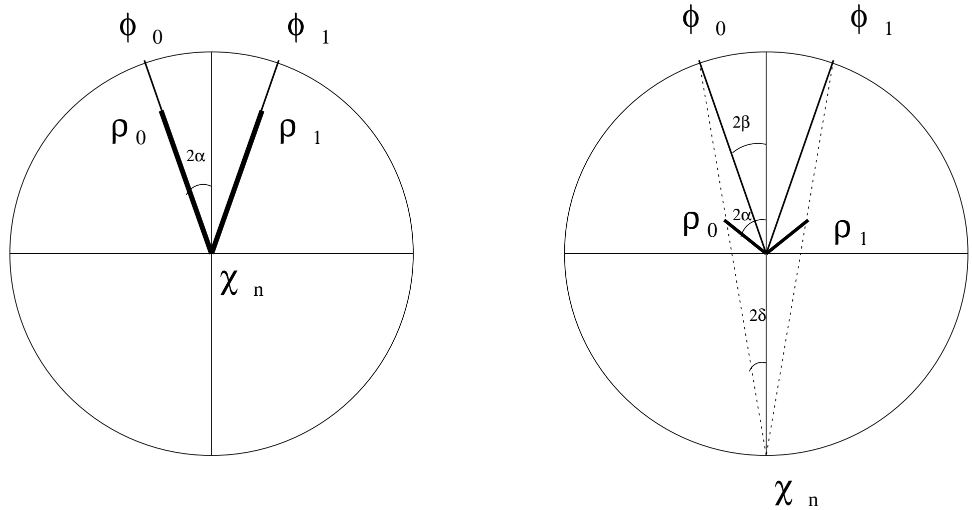
[11] This calculation was done with the help of D. Mayers, private communication.

[12] However, it is easy to check that α is of order ${p_e}^{1/2}$ if the EHPP attack is applied on various other states which are more sensitive to this specific eavesdropping attack. We emphasize this point to avoid the impression that the translucent attack we use later on is inferior to the EHPP attack.

[13] A. Peres, "Quantum Theory: Concepts and Methods", Kluwer Academic Publishers, Dordrecht (1993).

[14] C. A. Fuchs and A. Peres, *Phys. Rev. A.* **53**, 2038 (1996).

[15] For instance, let a singlet state be shared by Eve and Bob, and let Bob find his particle in a state $|\phi\rangle = \binom{\cos\eta}{\sin\eta}$. Replacing $\hat{I}$ by $|\phi\rangle\langle\phi|$ as in (5), we see that the state in Eve's hands is not ρ_{cms} but $\binom{\sin\eta}{-\cos\eta}$ as expected.



1 a **1 b**

Figure 1

Figure 1: Two ways of constructing the two density matrices ρ_p from two pure states Φ_p and a third state χ_n common to both density matrices. In a), $\chi_n = \rho_{cms}$, the completely mixed state. In b), $\chi_n = \downarrow_z$, the "down z" pure spin state.